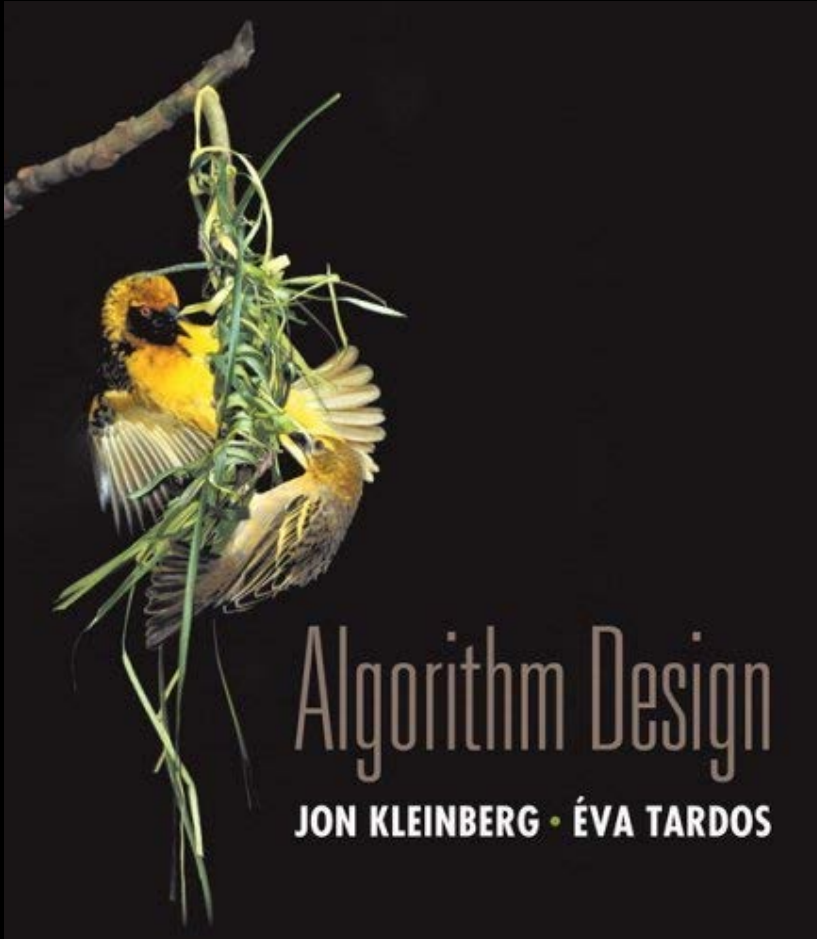


# Chapter 8

## NP and Computational Intractability



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## 8.3 Definition of NP

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# Decision Problems

Decision problem.

- $X$  is a set of strings.
- Instance: string  $s$ .
- Algorithm  $A$  solves problem  $X$ :  $A(s) = \text{yes}$  iff  $s \in X$ .

**Polynomial time.** Algorithm  $A$  runs in poly-time if for every string  $s$ ,  $A(s)$  terminates in at most  $p(|s|)$  "steps", where  $p(\cdot)$  is some polynomial.

↑  
length of  $s$

**PRIMES:**  $X = \{ 2, 3, 5, 7, 11, 13, 17, 23, 29, 31, 37, \dots \}$

**Algorithm.** [Agrawal-Kayal-Saxena, 2002]  $p(|s|) = |s|^8$ .

# Definition of P

P. Decision problems for which there is a poly-time algorithm.

Problem	Description	Algorithm	Yes	No
MULTIPLE	Is $x$ a multiple of $y$ ?	Grade school division	51, 17	51, 16
RELPRIME	Are $x$ and $y$ relatively prime?	Euclid (300 BCE)	34, 39	34, 51
PRIMES	Is $x$ prime?	AKS (2002)	53	51
EDIT-DISTANCE	Is the edit distance between $x$ and $y$ less than 5?	Dynamic programming	niether neither	acgggt tttta
LSOLVE	Is there a vector $x$ that satisfies $Ax = b$ ?	Gauss-Edmonds elimination	$\left[ \begin{array}{ccc c} 0 & 1 & 1 & 4 \\ 2 & 4 & -2 & 2 \\ 0 & 3 & 15 & 36 \end{array} \right]$	$\left[ \begin{array}{ccc c} 1 & 0 & 0 & 1 \\ 1 & 1 & 1 & 1 \\ 0 & 1 & 1 & 1 \end{array} \right]$

# NP

## Certification algorithm intuition.

- Certifier views things from "managerial" viewpoint.
- Certifier doesn't determine whether  $s \in X$  on its own; rather, it checks a proposed proof  $t$  that  $s \in X$ .

**Def.** Algorithm  $C(s, t)$  is a **certifier** for problem  $X$  if for every string  $s$ ,  $s \in X$  iff there exists a string  $t$  such that  $C(s, t) = \text{yes}$ .

↙  
"certificate" or "witness"

**NP.** Decision problems for which there exists a **poly-time** certifier.

↑  
 $C(s, t)$  is a poly-time algorithm and  
 $|t| \leq p(|s|)$  for some polynomial  $p(\cdot)$ .

**Remark.** NP stands for **nondeterministic** polynomial-time.

# Certifiers and Certificates: Composite

**COMPOSITES.** Given an integer  $s$ , is  $s$  composite?

**Certificate.** A nontrivial factor  $t$  of  $s$ . Note that such a certificate exists iff  $s$  is composite. Moreover  $|t| \leq |s|$ .

**Certifier.**

```
boolean C(s, t) {  
    if (t ≤ 1 or t ≥ s)  
        return false  
    else if (s is a multiple of t)  
        return true  
    else  
        return false  
}
```

**Instance.**  $s = 437,669$ .

**Certificate.**  $t = 541$  or  $809$ .  $\longleftarrow 437,669 = 541 \times 809$

**Conclusion.** COMPOSITES is in NP.

## Certifiers and Certificates: 3-Satisfiability

**SAT.** Given a CNF formula  $\Phi$ , is there a satisfying assignment?

**Certificate.** An assignment of truth values to the  $n$  boolean variables.

**Certifier.** Check that each clause in  $\Phi$  has at least one true literal.

**Ex.**

$$(\bar{x}_1 \vee x_2 \vee x_3) \wedge (x_1 \vee \bar{x}_2 \vee x_3) \wedge (x_1 \vee x_2 \vee x_4) \wedge (\bar{x}_1 \vee \bar{x}_3 \vee \bar{x}_4)$$

instance  $s$

$$x_1 = 1, x_2 = 1, x_3 = 0, x_4 = 1$$

certificate  $t$

**Conclusion.** SAT is in NP.

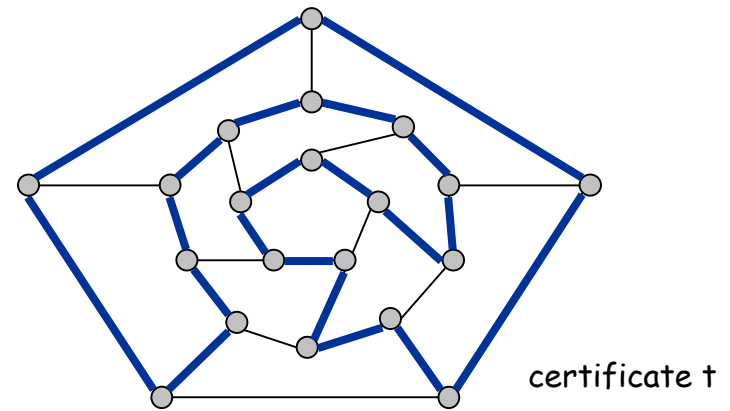
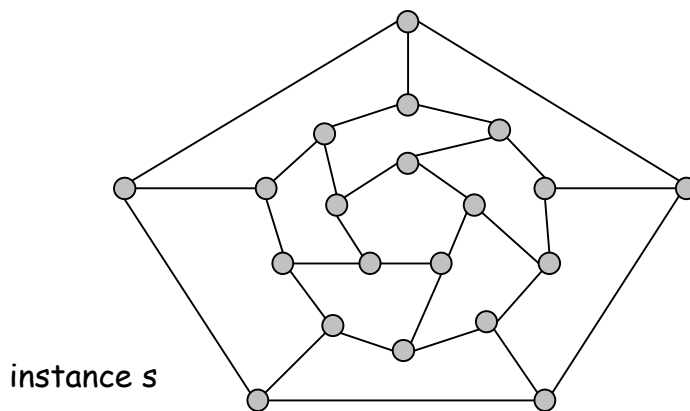
# Certifiers and Certificates: Hamiltonian Cycle

**HAM-CYCLE.** Given an undirected graph  $G = (V, E)$ , does there exist a simple cycle  $C$  that visits every node?

**Certificate.** A permutation of the  $n$  nodes.

**Certifier.** Check that the permutation contains each node in  $V$  exactly once, and that there is an edge between each pair of adjacent nodes in the permutation.

**Conclusion.** HAM-CYCLE is in NP.





# P, NP, EXP

**P.** Decision problems for which there is a **poly-time algorithm**.

**EXP.** Decision problems for which there is an **exponential-time algorithm**.

**NP.** Decision problems for which there is a **poly-time certifier**.

**Claim.**  $P \subseteq NP$ .

**Pf.** Consider any problem  $X$  in  $P$ .

- By definition, there exists a poly-time algorithm  $A(s)$  that solves  $X$ .
- Certificate:  $t = \varepsilon$ , certifier  $C(s, t) = A(s)$ . ▪

**Claim.**  $NP \subseteq EXP$ .

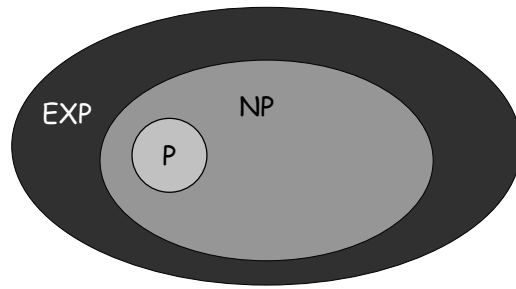
**Pf.** Consider any problem  $X$  in  $NP$ .

- By definition, there exists a poly-time certifier  $C(s, t)$  for  $X$ .
- To solve input  $s$ , run  $C(s, t)$  on all strings  $t$  with  $|t| \leq p(|s|)$ .
- Return  $_{yes}$ , if  $C(s, t)$  returns  $_{yes}$  for any of these. ▪

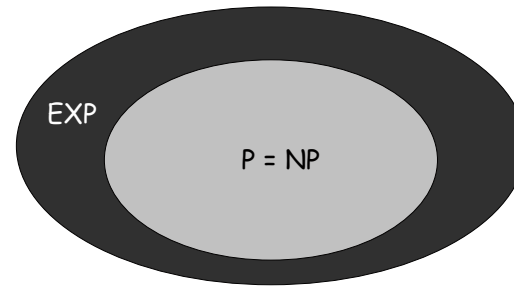
# The Main Question: P Versus NP

Does  $P = NP$ ? [Cook 1971, Edmonds, Levin, Yablonski, Gödel]

- Is the decision problem as easy as the certification problem?
- Clay \$1 million prize.



If  $P \neq NP$



If  $P = NP$

would break RSA cryptography  
(and potentially collapse economy)

If yes: Efficient algorithms for 3-COLOR, TSP, FACTOR, SAT, ...

If no: No efficient algorithms possible for 3-COLOR, TSP, SAT, ...

Consensus opinion on  $P = NP$ ? Probably no.

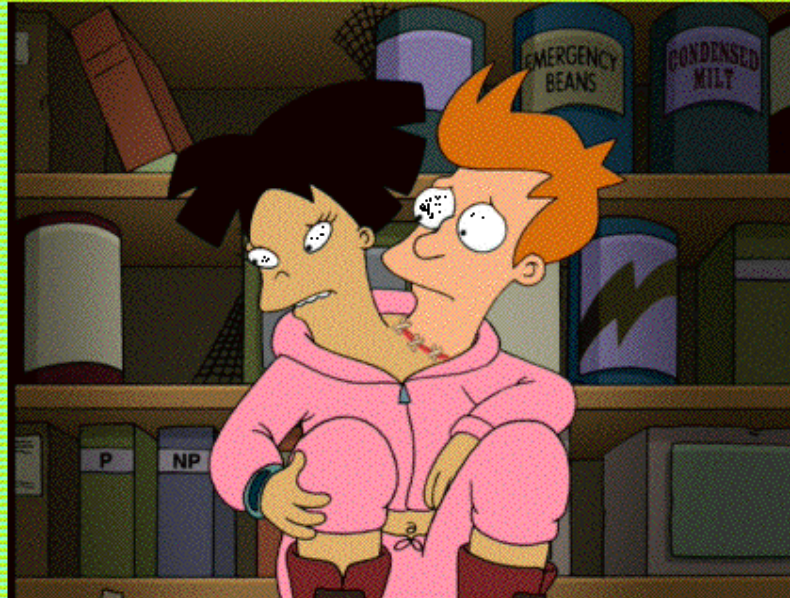
# The Simpson's: $P = NP$ ?



Copyright © 1990, Matt Groening

Futurama:  $P = NP?$

$P = NP ?$



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## Looking for a Job?

### Some writers for the Simpsons and Futurama.

- J. Steward Burns. M.S. in mathematics, Berkeley, 1993.
- David X. Cohen. M.S. in computer science, Berkeley, 1992.
- Al Jean. B.S. in mathematics, Harvard, 1981.
- Ken Keeler. Ph.D. in applied mathematics, Harvard, 1990.
- Jeff Westbrook. Ph.D. in computer science, Princeton, 1989.

## 8.4 NP-Completeness

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# Polynomial Transformation

**Def.** Problem  $X$  **polynomially reduces** (Cook) to problem  $Y$  if arbitrary instances of problem  $X$  can be solved using:

- Polynomial number of standard computational steps, plus
- Polynomial number of calls to oracle that solves problem  $Y$ .

**Def.** Problem  $X$  **polynomially transforms** (Karp) to problem  $Y$  if given any input  $x$  to  $X$ , we can construct an input  $y$  such that  $x$  is a  $yes$  instance of  $X$  iff  $y$  is a  $yes$  instance of  $Y$ .

↑  
we require  $|y|$  to be of size polynomial in  $|x|$

**Note.** Polynomial transformation is polynomial reduction with just one call to oracle for  $Y$ , exactly at the end of the algorithm for  $X$ . Almost all previous reductions were of this form.

**Open question.** Are these two concepts the same?

↑  
we abuse notation  $\leq_p$  and blur distinction

# NP-Complete

**NP-complete.** A problem  $Y$  in NP with the property that for every problem  $X$  in NP,  $X \leq_p Y$ .

**Theorem.** Suppose  $Y$  is an NP-complete problem. Then  $Y$  is solvable in poly-time iff  $P = NP$ .

**Pf.  $\Leftarrow$**  If  $P = NP$  then  $Y$  can be solved in poly-time since  $Y$  is in NP.

**Pf.  $\Rightarrow$**  Suppose  $Y$  can be solved in poly-time.

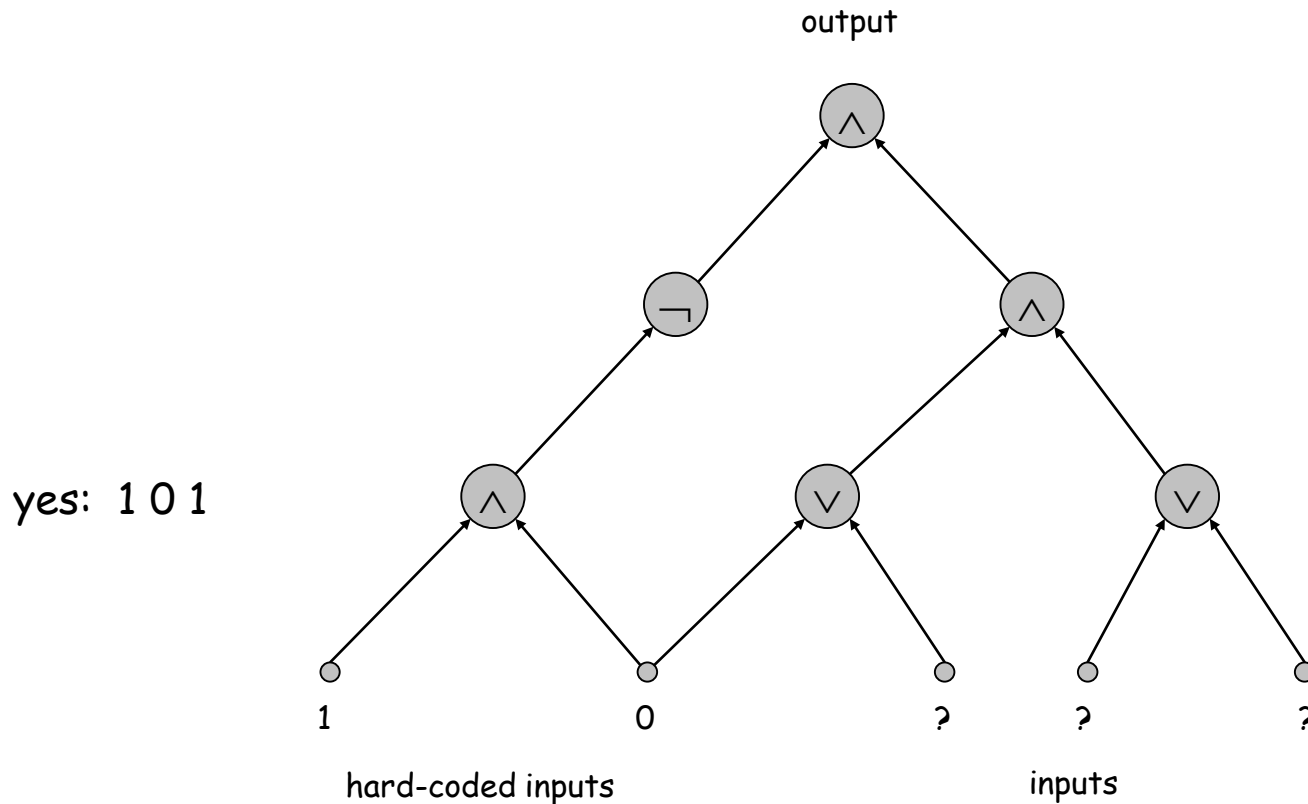
- Let  $X$  be any problem in NP. Since  $X \leq_p Y$ , we can solve  $X$  in poly-time. This implies  $NP \subseteq P$ .
- We already know  $P \subseteq NP$ . Thus  $P = NP$ . ▪

**Fundamental question.** Do there exist "natural" NP-complete problems?



# Circuit Satisfiability

**CIRCUIT-SAT.** Given a combinational circuit built out of AND, OR, and NOT gates, is there a way to set the circuit inputs so that the output is 1?



# The "First" NP-Complete Problem

**Theorem.** CIRCUIT-SAT is NP-complete. [Cook 1971, Levin 1973]

**Pf.** (sketch)

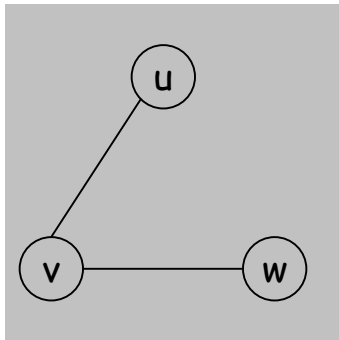
- Any algorithm that takes a fixed number of bits  $n$  as input and produces a yes/no answer can be represented by such a circuit. Moreover, if algorithm takes poly-time, then circuit is of poly-size.

sketchy part of proof; fixing the number of bits is important, and reflects basic distinction between algorithms and circuits

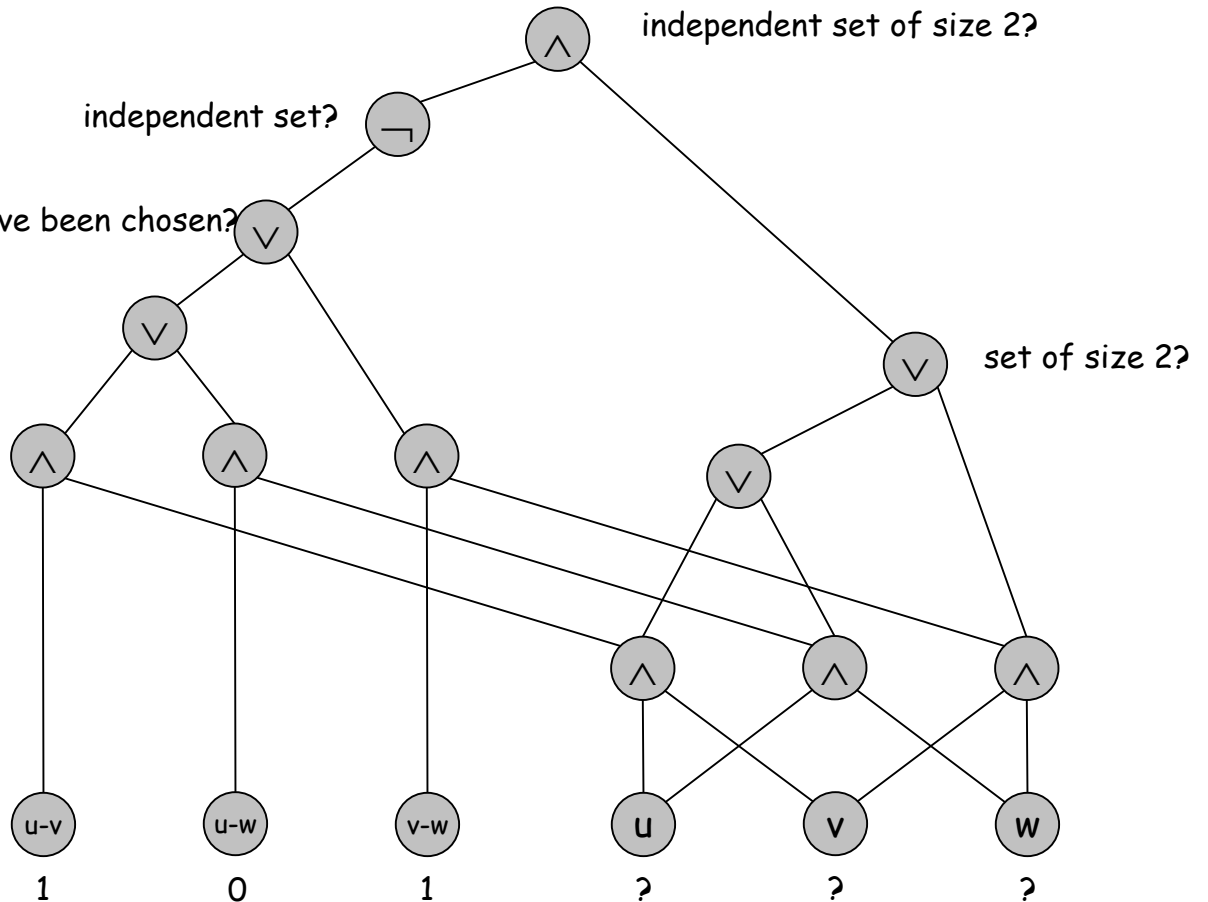
- Consider some problem  $X$  in NP. It has a poly-time certifier  $C(s, t)$ . To determine whether  $s$  is in  $X$ , need to know if there exists a certificate  $t$  of length  $p(|s|)$  such that  $C(s, t) = \text{yes}$ .
- View  $C(s, t)$  as an algorithm on  $|s| + p(|s|)$  bits (input  $s$ , certificate  $t$ ) and convert it into a poly-size circuit  $K$ .
  - first  $|s|$  bits are hard-coded with  $s$
  - remaining  $p(|s|)$  bits represent bits of  $t$
- Circuit  $K$  is satisfiable iff  $C(s, t) = \text{yes}$ .

# Example

Ex. Construction below creates a circuit  $K$  whose inputs can be set so that  $K$  outputs true iff graph  $G$  has an independent set of size 2.



$G = (V, E), n = 3$



$\binom{n}{2}$  hard-coded inputs (graph description)       $n$  inputs (nodes in independent set)

# Establishing NP-Completeness

**Remark.** Once we establish first "natural" NP-complete problem, others fall like dominoes.

**Recipe to establish NP-completeness of problem  $Y$ .**

- Step 1. Show that  $Y$  is in NP.
- Step 2. Choose an NP-complete problem  $X$ .
- Step 3. Prove that  $X \leq_p Y$ .

**Justification.** If  $X$  is an NP-complete problem, and  $Y$  is a problem in NP with the property that  $X \leq_p Y$  then  $Y$  is NP-complete.

**Pf.** Let  $W$  be any problem in NP. Then  $W \leq_p X \leq_p Y$ .

- By transitivity,  $W \leq_p Y$ .
- Hence  $Y$  is NP-complete. ▪

$\uparrow$                        $\uparrow$   
by definition of      by assumption  
NP-complete

# 3-SAT is NP-Complete

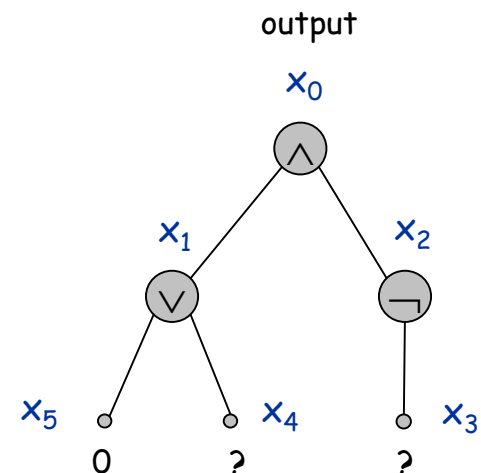
**Theorem.** 3-SAT is NP-complete.

**Pf.** Suffices to show that  $\text{CIRCUIT-SAT} \leq_p \text{3-SAT}$  since 3-SAT is in NP.

- Let  $K$  be any circuit.
- Create a 3-SAT variable  $x_i$  for each circuit element  $i$ .
- Make circuit compute correct values at each node:
  - $x_2 = \neg x_3 \Rightarrow$  add 2 clauses:  $x_2 \vee x_3, \overline{x_2} \vee \overline{x_3}$
  - $x_1 = x_4 \vee x_5 \Rightarrow$  add 3 clauses:  $x_1 \vee \overline{x_4}, x_1 \vee \overline{x_5}, \overline{x_1} \vee x_4 \vee x_5$
  - $x_0 = x_1 \wedge x_2 \Rightarrow$  add 3 clauses:  $\overline{x_0} \vee x_1, \overline{x_0} \vee x_2, x_0 \vee \overline{x_1} \vee \overline{x_2}$

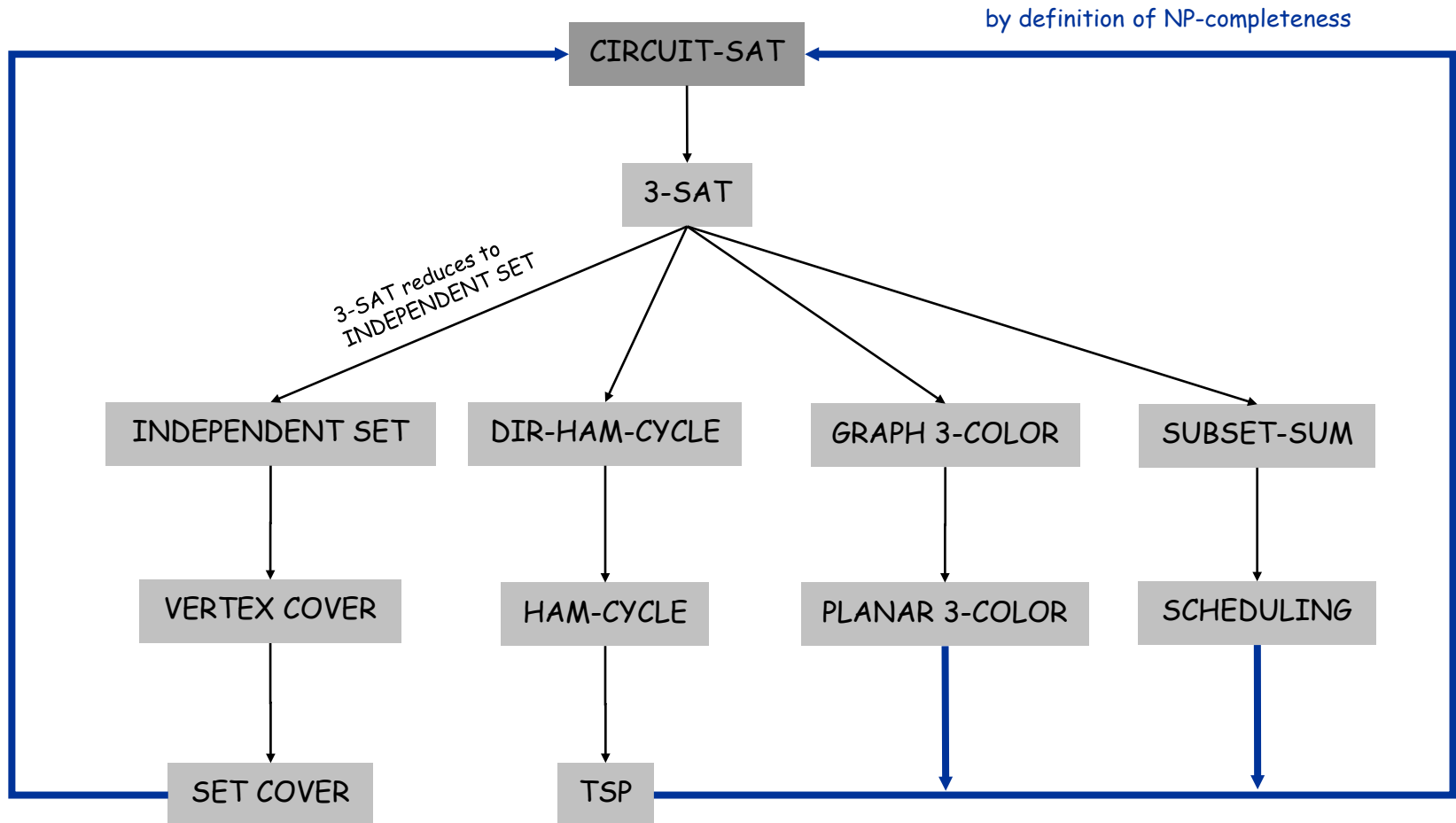
- Hard-coded input values and output value.
  - $x_5 = 0 \Rightarrow$  add 1 clause:  $\overline{x_5}$
  - $x_0 = 1 \Rightarrow$  add 1 clause:  $x_0$

- Final step: turn clauses of length  $< 3$  into clauses of length exactly 3. ▪



# NP-Completeness

**Observation.** All problems below are NP-complete and polynomial reduce to one another!



## Some NP-Complete Problems

Six basic genres of NP-complete problems and paradigmatic examples.

- Packing problems: SET-PACKING, INDEPENDENT SET.
- Covering problems: SET-COVER, VERTEX-COVER.
- Constraint satisfaction problems: SAT, 3-SAT.
- Sequencing problems: HAMILTONIAN-CYCLE, TSP.
- Partitioning problems: 3D-MATCHING 3-COLOR.
- Numerical problems: SUBSET-SUM, KNAPSACK.

**Practice.** Most NP problems are either known to be in P or NP-complete.

**Notable exceptions.** Factoring, graph isomorphism, Nash equilibrium.

# Extent and Impact of NP-Completeness

Extent of NP-completeness. [Papadimitriou 1995]

- Prime intellectual export of CS to other disciplines.
- 6,000 citations per year (title, abstract, keywords).
  - more than "compiler", "operating system", "database"
- Broad applicability and classification power.
- "Captures vast domains of computational, scientific, mathematical endeavors, and seems to roughly delimit what mathematicians and scientists had been aspiring to compute feasibly."

NP-completeness can guide scientific inquiry.

- 1926: Ising introduces simple model for phase transitions.
- 1944: Onsager solves 2D case in tour de force.
- 19xx: Feynman and other top minds seek 3D solution.
- 2000: Istrail proves 3D problem NP-complete.



# More Hard Computational Problems

- Aerospace engineering:** optimal mesh partitioning for finite elements.
- Biology:** protein folding.
- Chemical engineering:** heat exchanger network synthesis.
- Civil engineering:** equilibrium of urban traffic flow.
- Economics:** computation of arbitrage in financial markets with friction.
- Electrical engineering:** VLSI layout.
- Environmental engineering:** optimal placement of contaminant sensors.
- Financial engineering:** find minimum risk portfolio of given return.
- Game theory:** find Nash equilibrium that maximizes social welfare.
- Genomics:** phylogeny reconstruction.
- Mechanical engineering:** structure of turbulence in sheared flows.
- Medicine:** reconstructing 3-D shape from biplane angiogram.
- Operations research:** optimal resource allocation.
- Physics:** partition function of 3-D Ising model in statistical mechanics.
- Politics:** Shapley-Shubik voting power.
- Pop culture:** Minesweeper consistency.
- Statistics:** optimal experimental design.

## 8.9 co-NP and the Asymmetry of NP

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# Asymmetry of NP

**Asymmetry of NP.** We only need to have short proofs of *yes* instances.

**Ex 1.** SAT vs. TAUTOLOGY.

- Can prove a CNF formula is satisfiable by giving such an assignment.
- How could we prove that a formula is **not** satisfiable?

**Ex 2.** HAM-CYCLE vs. NO-HAM-CYCLE.

- Can prove a graph is Hamiltonian by giving such a Hamiltonian cycle.
- How could we prove that a graph is **not** Hamiltonian?

**Remark.** SAT is NP-complete and  $SAT \equiv_p TAUTOLOGY$ , but how do we classify TAUTOLOGY?

↑  
not even known to be in NP

## NP and co-NP

**NP.** Decision problems for which there is a poly-time certifier.

**Ex.** SAT, HAM-CYCLE, COMPOSITES.

**Def.** Given a decision problem  $X$ , its **complement**  $\overline{X}$  is the same problem with the yes and no answers reverse.

**Ex.**  $\overline{X} = \{ 0, 1, 4, 6, 8, 9, 10, 12, 14, 15, \dots \}$

$X = \{ 2, 3, 5, 7, 11, 13, 17, 23, 29, \dots \}$

**co-NP.** Complements of decision problems in NP.

**Ex.** TAUTOLOGY, NO-HAM-CYCLE, PRIMES.

## NP = co-NP ?

Fundamental question. Does NP = co-NP?

- Do  $\text{yes}$  instances have succinct certificates iff  $\text{no}$  instances do?
- Consensus opinion: no.

Theorem. If  $\text{NP} \neq \text{co-NP}$ , then  $\text{P} \neq \text{NP}$ .

Pf idea.

- P is closed under complementation.
- If  $\text{P} = \text{NP}$ , then NP is closed under complementation.
- In other words,  $\text{NP} = \text{co-NP}$ .
- This is the contrapositive of the theorem.

# Good Characterizations

Good characterization. [Edmonds 1965]  $NP \cap co-NP$ .

- If problem  $X$  is in both  $NP$  and  $co-NP$ , then:
  - for  $yes$  instance, there is a succinct certificate
  - for  $no$  instance, there is a succinct disqualifier
- Provides conceptual leverage for reasoning about a problem.

Ex. Given a bipartite graph, is there a perfect matching.

- If yes, can exhibit a perfect matching.
- If no, can exhibit a set of nodes  $S$  such that  $|N(S)| < |S|$ .

# Good Characterizations

**Observation.**  $P \subseteq NP \cap \text{co-NP}$ .

- Proof of max-flow min-cut theorem led to stronger result that max-flow and min-cut are in P.
- Sometimes finding a good characterization seems easier than finding an efficient algorithm.

**Fundamental open question.** Does  $P = NP \cap \text{co-NP}$ ?

- Mixed opinions.
- Many examples where problem found to have a non-trivial good characterization, but only years later discovered to be in P.
  - linear programming [Khachiyan, 1979]
  - primality testing [Agrawal-Kayal-Saxena, 2002]

**Fact.** Factoring is in  $NP \cap \text{co-NP}$ , but not known to be in P.

↑  
if poly-time algorithm for factoring,  
can break RSA cryptosystem

# PRIMES is in $NP \cap co-NP$

**Theorem.** PRIMES is in  $NP \cap co-NP$ .

**Pf.** We already know that PRIMES is in co-NP, so it suffices to prove that PRIMES is in NP.

**Pratt's Theorem.** An odd integer  $s$  is prime iff there exists an integer  $1 < t < s$  s.t.

$$t^{s-1} \equiv 1 \pmod{s}$$

$$t^{(s-1)/p} \not\equiv 1 \pmod{s}$$

for all prime divisors  $p$  of  $s-1$

**Input.**  $s = 437,677$

**Certificate.**  $t = 17, 2^2 \times 3 \times 36,473$



prime factorization of  $s-1$   
also need a recursive certificate  
to assert that 3 and 36,473 are prime

**Certifier.**

- Check  $s-1 = 2 \times 2 \times 3 \times 36,473$ .
- Check  $17^{s-1} \equiv 1 \pmod{s}$ .
- Check  $17^{(s-1)/2} \equiv 437,676 \pmod{s}$ .
- Check  $17^{(s-1)/3} \equiv 329,415 \pmod{s}$ .
- Check  $17^{(s-1)/36,473} \equiv 305,452 \pmod{s}$ .



use repeated squaring



## FACTOR is in $NP \cap co-NP$

**FACTORIZE.** Given an integer  $x$ , find its prime factorization.

**FACTOR.** Given two integers  $x$  and  $y$ , does  $x$  have a nontrivial factor less than  $y$ ?

**Theorem.**  $FACTOR \equiv_p FACTORIZE$ .

**Theorem.**  $FACTOR$  is in  $NP \cap co-NP$ .

**Pf.**

- **Certificate:** a factor  $p$  of  $x$  that is less than  $y$ .
- **Disqualifier:** the prime factorization of  $x$  (where each prime factor is less than  $y$ ), along with a certificate that each factor is prime.

# Primality Testing and Factoring

We established:  $\text{PRIMES} \leq_p \text{COMPOSITES} \leq_p \text{FACTOR}$ .

Natural question: Does  $\text{FACTOR} \leq_p \text{PRIMES}$  ?

Consensus opinion. No.

State-of-the-art.

- PRIMES is in P. ← proved in 2001
- FACTOR not believed to be in P.

RSA cryptosystem.

- Based on dichotomy between complexity of two problems.
- To use RSA, must generate large primes efficiently.
- To break RSA, suffices to find efficient factoring algorithm.

# Extra Slides

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# Princeton CS Building, West Wall





# Princeton CS Building, West Wall



Character	ASCII	Bits
P	80	1010000
=	61	0111101
N	78	1001110
P	80	1010000
?	63	0111111

# Not How To Give a PowerPoint Talk

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(commercial break)

## A Note on Terminology

Knuth. [SIGACT News 6, January 1974, p. 12 - 18]

Find an adjective  $x$  that sounds good in sentences like.

- EUCLIDEAN-TSP is  $x$ .
- It is  $x$  to decide whether a given graph has a Hamiltonian cycle.
- It is unknown whether FACTOR is an  $x$  problem.

**Note:**  $x$  does not necessarily imply that a problem is in NP, just that every problem in NP polynomial reduces to  $x$ .

# A Note on Terminology

## Knuth's original suggestions.

- Hard.
- Tough.
- Herculean. but Hercules known for strength not time
- Formidable.
- Arduous.

## Some English word write-ins.

- Impractical.
- Bad.
- Heavy.
- Tricky.
- Intricate.
- Prodigious.
- Difficult.
- Intractable.
- Costly.
- Obdurate.
- Obstinate.
- Exorbitant.
- Interminable.



## A Note on Terminology

Hard-boiled. [Ken Steiglitz] In honor of Cook.

Hard-ass. [Al Meyer] Hard as satisfiability.

Sisyphean. [Bob Floyd] Problem of Sisyphus was time-consuming.

but Sisyphus never finished his task

Ulyssean. [Don Knuth] Ulysses was known for his persistence.

and finished!

## A Note on Terminology: Made-Up Words

**Supersat.** [Al Meyer] Greater than or equal to satisfiability.

**Polychronious.** [Ed Reingold] Enduringly long; chronic.

like today's lecture

**PET.** [Shen Lin] Probably exponential time.

depending on P=NP conjecture: provably exponential time,  
or previously exponential time

**GNP.** [Al Meyer] Greater than or equal to NP in difficulty.

costing more than GNP to resolve

## A Note on Terminology: Consensus

**NP-complete.** A problem in NP such that every problem in NP polynomial reduces to it.

**NP-hard.** [Bell Labs, Steve Cook, Ron Rivest, Sartaj Sahni]  
A decision problem such that every problem in NP reduces to it.

not necessarily in NP

**NP-hard search problem.** A problem such that every problem in NP reduces to it.

not necessarily a yes/no problem

"creative research workers are as full of ideas for new terminology as they are empty of enthusiasm for adopting it." -Don Knuth